Discrete Mathematics 82 (1990) 233–250
North-Holland

DRAWING ORDERS WITH FEW SLOPES

Jurek CZYZOWICZ*◊, Andrzej PELC*† and Ivan RIVAL**‡
*Département d'Informatique, Université du Québec à Hull, C.P. 1250, Succ. "B", Hull, Québec J8X 3X7, Canada
**Department of Computer Science, University of Ottawa, Ottawa, Ontario K1N 9B4, Canada

Received 16 April 1987
Revised 1 June 1988

It is common to draw a diagram of an ordered set with as few slopes as seem possible; the maximum number of upper covers or lower covers of an element is an obvious lower bound to the number of different slopes needed. We construct lattices with at most two (respectively, three) upper and lower covers which require at least three (respectively, four) different slopes – despite a conjecture of B. Sands to the contrary. Moreover, we characterize lattices with two-slope diagrams. It follows, for example, that every planar lattice with at most two upper and two lower covers has a (planar) two-slope diagram.

1. Introduction

Precedence relations due to technological constraints on an underlying set of jobs generate order in data structures. The actual presentation of these data structures may, of course, play an important role in computations, and even in decision-making. By far the most common presentation of ordered sets is the graphical representation scheme called the "diagram".

For elements a and b in an ordered set P say that *a covers b* or *b* is *covered by a*, and write $a \succ b$ if $a > b$ and, for each x in P, $a > x \geqslant b$ implies $x = b$. We also call a an *upper cover* of b, b a *lower cover* of a, and $\{a, b\}$ a *covering pair*. A *diagram* of P is a pictorial representation of P in the plane in which small circles, corresponding to the elements of P, are arranged in such a way that, for a and b in P, the circle corresponding to a is higher than the circle corresponding to b whenever $a > b$ and a straight line segment is drawn to connect the two circles just if a covers b. Insofar as a diagram is, in the end, a "drawing" of an ordered set, there is considerable variation possible in the actual rendering – despite the fact that any diagram determines its ordered set.

What are the criteria for a "good" diagram? For one thing the edges in a diagram are usually drawn as "steep" as possible to emphasize the ordering relation. For instance, a *covering chain*, that is, a sequence of successive covering pairs of the elements in a chain, may often be drawn on a vertical line. (Indeed, according to its definition no diagram can use any horizontal lines at all.) On the

◊ Partially supported by NSER grant OGP23992.
† Partially supported by NSERC grant A8136.
‡ Partially supported by NSERC grant A4077.

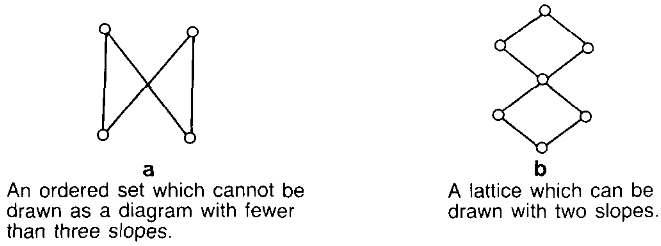


Fig. 1.

other hand, it is an everyday inclination common to all, who have experienced the preparation of many such diagrams for display, to minimize the actual number of slopes needed. This paper is inspired by a conjecture tentatively put forth by B. Sands in early 1984 [9] that the minimum number of slopes needed to draw a lattice depends just on the maximum number of upper covers and of lower covers among the elements of the lattice, that is, the maximum *up-degree* and *down-degree* of an element. It is obvious that, for any ordered set, the minimum number of slopes needed is at least the maximum of the up-degrees and down-degrees of its elements. Sands conjectured that this is precisely the number needed – as long as the ordered set is a lattice (cf. *Fig. 1*). Duffus [3] fueled this conjecture by observing that every finite distributive lattice can be drawn with this number of slopes. His observation was based on these two facts: every finite distributive lattice with maximum up-degree and down-degree k can be embedded, as a cover-preserving sublattice, in the direct product of k chains; a direct product of k chains (like the k-dimensional hypercube) can be drawn with precisely k slopes.

This evidence notwithstanding, the conjecture is false. The lattice illustrated in *Fig. 2a* has maximum up-degree and down-degree two, yet, as we will see, it cannot be drawn with fewer than three slopes. The lattice illustrated in *Fig. 2b* has maximum up-degree and down-degree three, yet it cannot be drawn with fewer than four slopes.

On the other hand, let L be a lattice with a *two-slope diagram,* that is, the

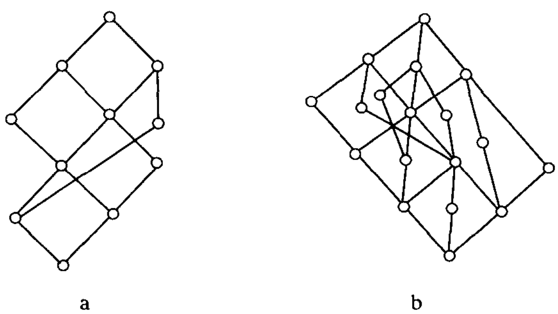


Fig. 2.

edges of this diagram use only two slopes. We may assume, without any loss in generality, that the corresponding angles are 45° and 135°. It is convenient to associate a colour to each of the covering pairs of elements $\{x, y\}$ in such a way that the covering pair $\{x, y\}$ is red, if $x \prec y$ and the edge from x to y in the diagram makes an angle of 45° with the horizontal, and the covering pair $\{x, y\}$ is blue, if $x \prec y$ and the edge from x to y in the diagram makes 135° with the horizontal. If $a \prec b$ and $a \prec c$ or $a \succ b$ and $a \succ c$ then, evidently, the covering pairs $\{a, b\}$ and $\{a, c\}$ are assigned different colours. Suppose that there are two vertices $a > b$ which, in the two-slope diagram are joined by a covering chain each covering pair of which is assigned the same colour, red say. Suppose, moreover, that there is another covering chain joining a and b, all of whose covering pairs are distinct from the first. Let $a \succ c \geq d \succ b$ in the second chain. Obviously, the edges joining b to d and c to a are on opposite sides of the first chain and c cannot then be joined to d by a covering chain using only 45° and 135° angles (see *Fig. 3*).

Thus, it is easy to see that any monochromatic covering chain joining a and b, in a two-slope diagram, must be the only covering chain between a and b. That these conditions are sufficient too is the substance of our principal result.

Two-slope theorem. *A finite lattice has a two-slope diagram if and only if there is a two-colouring of all covering pairs of vertices according to which*

(i) *pairs $\{a, b\}$, $\{a, c\}$ are assigned different colours whenever a is covered by both b and c or a covers both b and c, and,*

(ii) *if $a > b$ and there is a monochromatic covering chain between them then there is no other covering chain between them.*

Besides the case of distributive lattices there is at least one other important class of lattices each of which has a two-slope diagram.

Corollary. *Every finite planar lattice with maximum up-degree and down-degree two, has a two-slope diagram.*

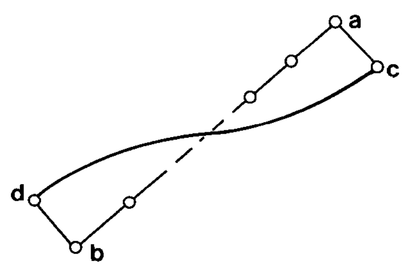


Fig. 3.

In fact, as we shall see, too, *every finite planar lattice with maximum up-degree and down-degree two has a planar, two-slope diagram.*

2. Two-slope drawings: the general case

The proof of the Two-Slope Theorem involves an analysis of elements of degree two, dismantlability and cycles in lattices. We deal with these items in turn. The first two, at least, are of independent interest.

Let L be a finite lattice with at least two elements. Let t stand for the top element and b for the bottom element. Thus, every element x satisfies $b \leqslant x \leqslant t$. An element x satisfying $b < x < t$ has *degree two* just if it has precisely one lower cover $\underset{\sim}{x}$ and precisely one upper cover $\bar{x}$. Let $D(L)$ stand for the set of all such elements of the lattice L. Say that L is *dismantlable* if its elements can be enumerated $L = \{x_1, x_2, \ldots, x_n\}$ in such a way that, for each $i \leqslant n-2$,

$$x_i \in D(L - \{x_1, x_2, \ldots, x_{i-1}\}),$$

$x_{n-1} = b$, and $x_n = t$. Thus, a dismantlable lattice can be decomposed, one element at a time, into a succession of sublattices, each with one less element than before, arriving finally at the two-element sublattice $\{b < t\}$ (*Fig. 4*). Notice too that the covering edges of any sublattice so obtained need not be actual covering edges of L. Thus, in L_1, $x_3 \succ\!- x_4$ although not in L_0; in L_5, $x_8 \succ\!- x_6$ although not in L_6 (and hence not in $L = L_0$) and, of course, $t = x_8 \succ\!- x_7 = b$ at the every last step L_6, although not at any preceding stage.

Dismantlable lattices have been extensively studied (cf. [1, 2, 5, 6, 8]). Perhaps the best known examples of dismantlable lattices are *planar* lattices, that is,

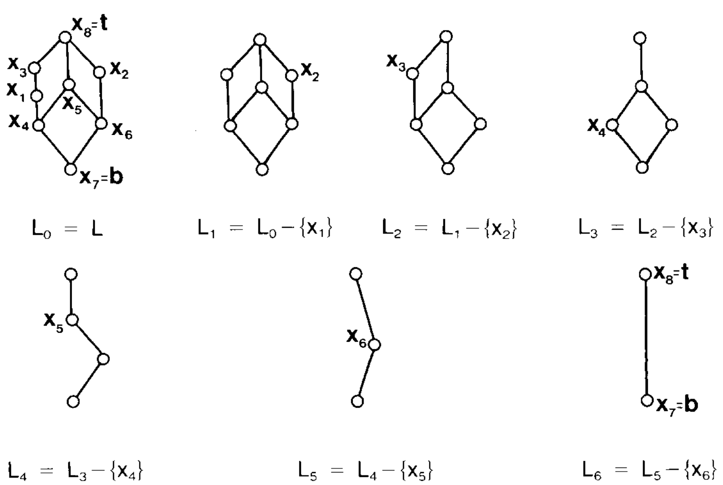


Fig. 4.

lattices for which there is a diagram in which no line segments cross, except possibly at their endpoints, where they may meet in a common element. As a matter of fact, there is always an element of degree two on the left boundary of any planar lattice (cf. [2, 6]).

The next proposition which is central to the proof of our principal result seems to be of interest on its own.

Proposition 1. *A finite lattice L with at least two elements is dismantlable if and only if there is a partition $C_1, C_2, \ldots, C_k$ of the covering pairs of L such that, for each $j \geq 1$, C_j is a covering chain of L each of whose internal vertices has degree two in $C_1 \cup C_2 \cup \cdots \cup C_j$, and b, t are vertices in C_1.*

We may reconstruct the sequence $C_k, C_{k-1}, \ldots, C_1$ in reverse order and so the "dismantling" of L can be carried out in such a way that, at each stage, a sublattice is obtained whose diagram is indeed a subdiagram of the original diagram and, which is itself obtained by removing a chain of elements each of degree two, at that stage (see *Fig. 5*).

Proof of Proposition 1. As each C_i consists just of degree two elements, at that stage, it is obvious that the conditions guarantee the dismantlability of L.

To construct C_1 we identify its elements, in reverse order from the dismantling sequence for L. To begin $x_n = t$, $x_{n-1} = b$ and $x_{n-1} = b \leq x_{n-2} \leq t = x_n$ shall all belong to C_1. If these three elements already satisfy $x_{n-1} \prec x_{n-2} \prec x_n$ then put $C_1 = \{x_{n-1} < x_{n-2} < x_n\}$. Otherwise, choose the largest index $i_1 < n-2$ such that x_n, x_{n-1}, x_{n-2}, x_{i_1} are all comparable in L. If this is not yet a covering chain choose the largest index $i_2 < i_1$ such that x_n, x_{n-1}, x_{n-2}, x_{i_1}, x_{i_2} are all comparable in L. Continue in this way until a maximal covering chain is constructed and call it C_1. This initial covering chain, which is a maximal chain in L, satisfies the required conditions.

Next we shall construct another dismantling sequence from $x_1, x_2, \ldots, x_n$ in such a way that the vertices of C_1 constitute a final segment of the new

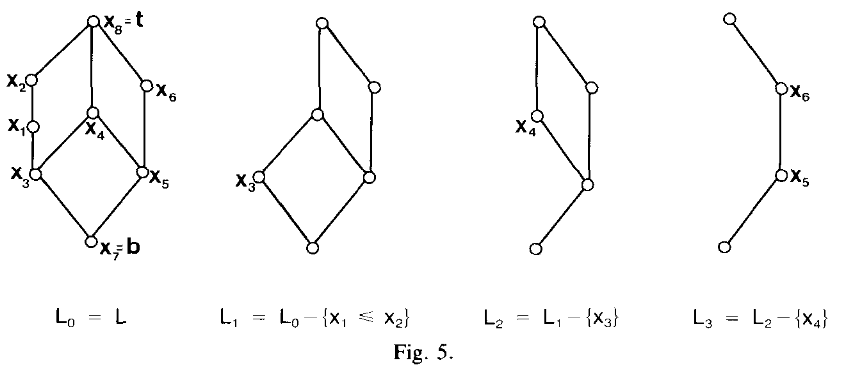


Fig. 5.

dismantling sequence. To this end let i_j be the largest index such that x_{i_j+1} is not a vertex in C_1, although x_{i_j} is a vertex in C_1. According to its construction $x_n, x_{n-1}, x_{n-2}, x_{i_1}, x_{i_2}, \ldots, x_{i_j}$ is a chain and this chain must, in fact, contain the unique upper cover $\overline{x_{i_j}}$ and the unique lower cover $\underline{x_{i_j}}$ of x_{i_j} in $L-\{x_1, x_2, \ldots, x_{i_j-1}\}$. In particular, neither $x_{i_j} \prec x_{i_j+1}$ nor $x_{i_j+1} \prec x_{i_j}$ which, in turn, implies that

$$x_1, x_2, \ldots, x_{i_j-1}, x_{i_j+1}, x_{i_j}, x_{i_j+2}, \ldots, x_{n-1}, x_n$$

is also a dismantling sequence for L. We continue in this fashion until we have constructed another dismantling sequence for L in which all vertices of C_1 do occur as a final segment. Once done, C_2 is constructed next. With respect to the current dismantling sequence choose the largest index i_1 such that x_{i_1} is not a vertex in C_1. Let b_1 be the largest vertex of C_1 below x_{i_1} and let t_1 be the smallest vertex of C_1 above x_{i_1}. If the sequence $(\{b, x_{i_1}\}, \{x_{i_1}, t\})$ is a covering chain of L, call it C_2. Otherwise, choose the largest index $i_2 < i_1$ such that the elements t_1, b_1, x_{i_1}, x_{i_2} are comparable in L and lie between t_1 and b_1. Continue in this way until a covering chain is constructed and call it C_2. We can again rearrange the dismantling sequence up to x_{i_1} so that all new vertices of C_2 form a string just before those from C_1. This is done as before by transposing successively consecutive pairs of elements, the first of which belongs to C_2 and the second, its successor, does not. Because such a pair will not be in any covering relation, the one with the other, these successive interchanges may be made, producing another dismantling sequence.

The process may then be repeated until all covering pairs of L are assigned to some such "assembling" chain. The construction, too, guarantees that these chains C_j do, at each stage, satisfy our conditions. □

Before we come directly to the proof of the Two-Slope Theorem we require an analysis of the relation between cycles and the degree of elements.

For an integer $n \geq 3$, a *cycle* $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ is an ordered set in which $x_i < y_i$, $x_{i+1} < y_i$, for $i = 1, 2, \ldots, n-1$, $x_1 < y_n$ are the only comparabilities (see *Fig. 6*).

An important characterization is this. *A finite lattice is dismantlable if and only if it contains no cycles* ([1, 5]). We shall need it especially in the light of the next result which is, actually, implicit in [1].

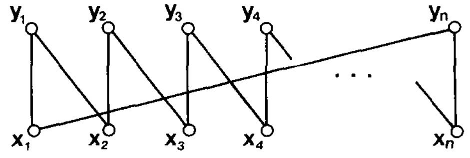


Fig. 6.

Lemma 1. *If L is a finite lattice whose shortest cycle has $2n$ elements, $n \geqslant 3$, then L contains an element of up-degree at least n and an element of down-degree at least n.*

Proof. Let $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ be a shortest cycle in L. We show, first of all, that there is then a cycle in L of the same size in which, the y_i's are the suprema of successive x_j's and the x_i's are infima of successive y_j's. To this end for each $i = 1, 2, \ldots, n-1$, let $y_i' = x_i + x_{i+1}$ and $y_n' = x_1 + x_n$. Then $x_i < y_i'$, $x_{i+1} < y_i'$, $y_i' < y_i$ and $\{x_1, y_1', x_2, y_2', \ldots, x_n, y_n'\}$ is a cycle. Next, for $i = 2, 3, \ldots, n$, set $x_i' = y_{i-1}' \cdot y_i'$ and $x_1' = y_1' \cdot y_n'$. Then $\{x_1', y_1', x_2', y_2', \ldots, x_n', y_n'\}$ is a cycle and, moreover, $x_i' + x_{i+1}' = y_i'$ and $x_1' + x_n' = y_n'$. Thus, without loss of generality, we may suppose that $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ is a shortest cycle in L in which, too, for each $i \leqslant n-1$, $y_i = x_i + x_{i+1}$, $y_n = x_1 + x_n$ and, for each $i \geqslant 2$, $x_i = y_{i-1} \cdot y_i$, $x_1 = y_1 \cdot y_n$.

Next, consider the set of all pairwise suprema $y_i + y_j$, $i \neq j$, and, choose one, say, $y_i + y_j$, $i \leqslant j$, which is minimal from among them. If $y_i + y_j$ were noncomparable with some y_k, $k \neq i, j$, then there would be a shorter cycle containing y_k and $y_i + y_j$.

Therefore, $y_i + y_j > y_k$ for all $k = 1, 2, \ldots, n$. According to its minimality, $y_i + y_j$ must, in fact, be the suprema of every distinct pair of y_k's. It follows, then, that $y_i + y_j$ must have n lower covers.

A dual argument shows that a maximal infimum $x_i \cdot x_j$ will have n upper covers too. □

Corollary. *Every finite lattice which contains a cycle contains an element of up-degree at least three and an element of down-degree at least three.*

A lattice may contain cycles of arbitrary size and yet have maximum up-degree and down-degree three (cf. *Fig. 7*).

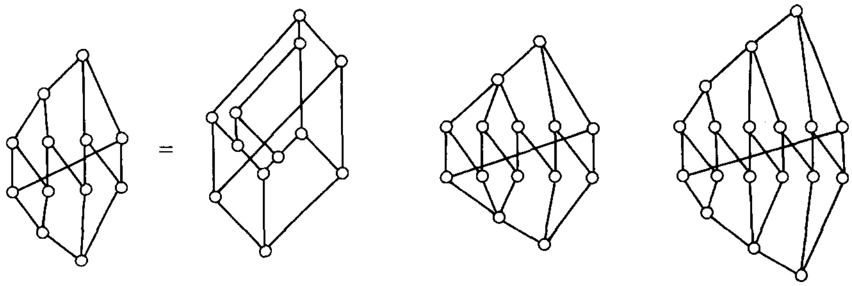


Fig. 7.

Proof of the Two-Slope Theorem

Conditions (i) and (ii) are necessary for the existence of a two-slope diagram, as we have already shown in the Introduction. We now prove their sufficiency.

Let L be a finite lattice for which there is a two-colouring, say red and blue, of all covering pairs which satisfies (i) and (ii). From (i) it follows that every vertex has at most two upper covers and at most two lower covers. Then, according to Lemma 1, L will not contain any cycles at all and hence it must be a dismantlable lattice. We may thus apply Proposition 1 according to which there is a sequence $C_1, C_2, \ldots$ of covering chains of L which, loosely speaking may be used to reconstruct L one chain at a time.

We construct a two-slope diagram of L by plotting the covering chains $C_1, C_2, \ldots,$ one by one, and tracing their respective edges. At each stage of the construction we preserve two conditions:

(*) edges joining red pairs have angle 45° and edges joining blue pairs have angle 135°;

(**) any two vertices on a 45° line or on a 135° line must be joined by a covering chain.

Suppose by induction that covering chains $C_1, \ldots, C_{i-1}$ are already plotted according to these conditions. Let $a > b$ be two elements comparable in $C_1 \cup \cdots \cup C_{i-1}$ which are endpoints of C_i. We distinguish two cases.

Case 1. The chain C_i has length two (see Fig. 8).
The vertices a and b have two distinct covering chains joining them in the lattice L, hence the pairs $\{a, u\}$ and $\{b, u\}$ cannot have the same colour. Without loss of generality we may suppose that $\{a, u\}$ is blue and $\{b, u\}$ is red. Let d be the other lower cover of a and c the other upper cover of b. According to condition (i) the pair $\{a, d\}$ is red and the pair $\{b, c\}$ is blue. Therefore, the edge representing $\{a, d\}$ has angle 45° and the edge representing $\{b, c\}$ has angle 135°. Since u can be the only other lower cover of a and the only other upper cover of b, condition (ii) implies that no other vertex is plotted on the half-line of angle 135° below a or on the half-line of angle 45° above b. It follows that we can plot u at the intersection of those half-lines, trace edges corresponding to covering pairs $\{a, u\}$ and $\{b, u\}$, thus preserving both conditions (i) and (ii).

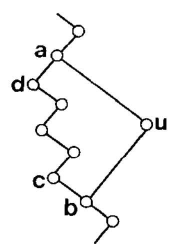


Fig. 8.

Case 2. The chain C_i has length greater than two (see Fig. 9).
Let $r_1, \ldots, r_k$ be all of the red covering pairs and $b_1, \ldots, b_m$ be all of the blue covering pairs listed according to their place in the chain C_i. Without loss of generality we may assume that the vertices of C_i are listed in decreasing order. Let a', b' be orthogonal projections of vertices a and b on the line $y = x$ and a'', b'' orthogonal projections of vertices a and b on the line $y = -x$. First notice that $a' \neq b'$ and $a'' \neq b''$. Otherwise, a and b would be on the same line of angle 135° or 45°. However, those vertices are already joined by a sequence of edges of angles 45° or 135°. It follows that all these edges must in fact have the same slope, that is, there is a monochromatic chain in $C_1 \cup \cdots \cup C_{i-1}$ joining a and b which, in view of condition (ii) contradicts the fact that C_i also joins these vertices.

We take a partition of the segment $[a', b']$ into k subintervals such that partitioning points are different from the orthogonal projections of all vertices on the line $y = x$. Let $t_1, \ldots, t_k$ be the lengths of intervals in this partition, starting from a'. Likewise, we take a partition of the segment $[a'', b'']$ into m subintervals such that partitioning points are different from orthogonal projections of all vertices on the line $y = -x$. Let $s_1, \ldots, s_m$ be the lengths of intervals in this partition starting from a''.

Now we can plot the chain C_i in a simple way. Consider consecutive covering pairs starting from the endpoint a. If the current pair is red represent it by a segment of angle 45° whose top endpoint is the bottom endpoint of the previous edge and whose length is the consecutive number t_j. If the pair is blue proceed similarly using angle 135° and a number s_j. Condition (i) allows us to construct the first and last edges. All other edges can be constructed in view of condition (**) which holds by induction. It is easy to see that both conditions (*) and (**) are preserved after adding the chain C_i to our diagram. Hence induction can be carried out and the proof is complete. □

The Two-Slope Theorem gives a simple way to check that the condition

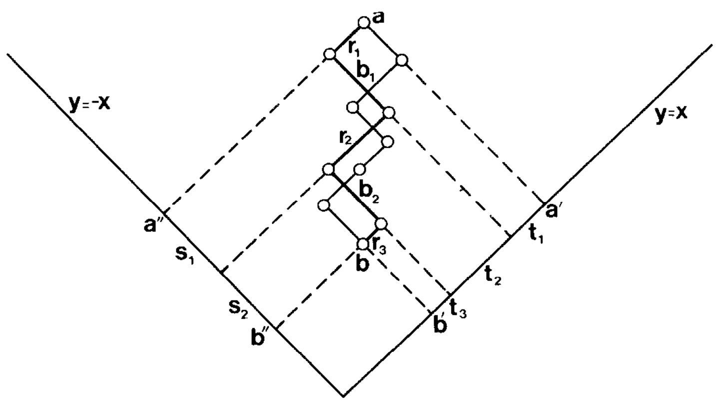


Fig. 9.

Fig. 10.

bounding the maximum up-degree and down-degree of the lattice by two is not enough to guarantee a two-slope diagram, contrary to Sands' conjecture. The two-colouring conditions imply, in particular, that, for a configuration as illustrated in *Fig. 10a* covering pairs A and B must have the same colour, different from that of A' and B'. *Fig. 10b* shows the simplest of such configurations. Hence for many lattices we can "force" several covering pairs to have the same colour. It sufficces to produce a lattice of maximum up-degree and down-degree two with a monochromatic covering chain (with respect to any two-colouring satisfying condition (i)), whose largest vertex has only one lower cover, whose smallest vertex has only one upper cover and to join those endpoints by an arbitrary new covering chain. This will contradict condition (ii) which, in turn, means that no two-slope diagram of the lattice exists although the requirement about the maximum up-degree and down-degree remains satisfied.

Perhaps the simplest counterexample to Sands' conjecture is the lattice illustrated again in *Fig. 11*.

In view of the property described on *Fig. 10b* it is obvious that, for any two-colouring satisfying the condition (ii) of the Two-Slope Theorem the covering pairs represented by edges A, B and C must have the same colour, whence the contradiction.

Of course for this simple lattice it is fairly easy to check "by hand" that it cannot have a two-slope diagram anyway. For more complicated lattices, however, our criterion seems often to be a convenient way to proceed. For instance, does the lattice illustrated in *Fig. 12* have a two-slope diagram?

Let us also note that our characterization of lattices with two-slope diagrams fails for arbitrary ordered sets. It is easy to see that no cycle can have a two-slope diagram although it clearly has a two-colouring as described in the Two-Slope Theorem.

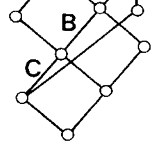


Fig. 11.

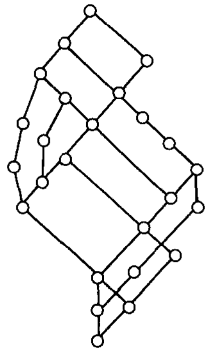


Fig. 12.

While out techniques here seem limited to the case of two-slope diagrams we conjecture the existence of a full range of such maximum up-degree and down-degree counterexamples.

Conjecture I. For every $n > 1$ there is a lattice with maximum up-degree and down-degree n which has no diagram using n slopes.

It is more complicated to verify that the lattice, illustrated before in *Fig. 2b*, cannot be drawn with only three slopes. Any case beyond $n = 3$ would seem to require a new idea.

Proposition 2. *There exists a lattice with maximum up-degree and down-degree three but which does not have a diagram using only three slopes.*

We sketch a proof that the lattice L illustrated in *Fig. 2b* (on *Fig. 13* its vertices are labelled with numbers used in the proof) does not have a diagram using only three slopes. Indeed, suppose it has. Without loss of generality we may assume that the angles are 135°, vertical, and 45°.

Notice that the lattice consisting of five elements: the top, the bottom and three pairwise noncomparable elements in between must have a diagram as in *Fig. 14*, where four vertices form a square and the fifth is on the vertical diagonal. Hence the part of the diagram of L generated by vertices 1, 2, 3, 4, 8, 10, 11, 12, 15 must be of the form as shown in *Fig. 15* where the relative positions of vertices 2, 3, 4 and 10, 11, 12 are yet to be determined.

It can be checked that vertex 12 from *Fig. 13* cannot be placed as x is on *Fig. 15*.

By symmetry we may assume that 12 is z and hence the part of the diagram of

L generated by the vertices 1, 2, 3, 4, 8, 10, 11, 12, 15, 16, 17, 18 must have the form shown on *Fig. 16.*

Vertex 12 must now be connected by new covering chains of length two with two of the vertices a, b, c. The only possible way to make these connections is with vertices b and c. Now one of the vertices d or e must be connected with vertex a by two distinct covering chains of length two, which is obviously impossible.

Therefore, L cannot have a three-slope diagram.

In spite of these examples and the Conjecture I above that there are more, we

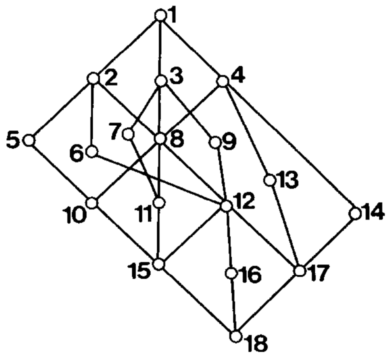


Fig. 13.

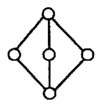


Fig. 14.

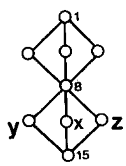


Fig. 15.

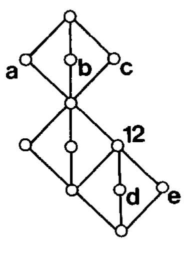


Fig. 16.

tentatively propose this natural variation on the original "degree-slope" conjecture.

Conjecture II. For every positve integer n there is a function $f(n)$ such that every finite lattice (even ordered set) with maximum up-degree and down-degree n has a diagram requiring at most $f(n)$ slopes.

For instance, what is $f(2)$? We cannot even decide whether $f(2) = 3$.

3. Planar lattices

Here is a simple consequence of the Two-Slope Theorem.

Corollary. *Every finite planar lattice with maximum up-degree and down-degree two has a two-slope diagram.*

Proof. We shall verify the conditions of the Two-Slope Theorem for a finite planar lattice L. Let a planar diagram of L be given. For any element with two upper covers colour the left edge blue and the right edge red and, for any element with two lower covers colour the left edge red and the right edge blue. As this colouring is done with respect to a fixed planar diagram no edge is assigned both red and blue. All uncoloured edges may now be coloured arbitrarily. We need only check that there is no monochromatic covering chain joining a comparable pair $a > b$ of vertices between which there is another covering chain. For contradictions suppose there is such a monochromatic blue chain from a to b. Then there is a lower cover c of a whose corresponding edge is coloured red, whence c lies to the left of the other lower cover of a and, there is an upper cover d of b such that the edge joining d to b is coloured red, whence d lies to the right of the other upper cover of b. Finally, $c > d$ which implies that any covering chain from c to d must cross the blue chain from a to b. Then there is an element e on the blue chain with two lower covers, one on the blue chain, to the left of the other which lies on the chain joining c and d. This, however, contradicts the rules according to which the colours were assigned in the first place.

We may well ask whether a planar lattice with maximum up-degree and down-degree two has a two-slope diagram which is, in addition, planar? The Corollary above, as seen through the Two-Slope Theorem, does not seem to guarantee it (see *Fig. 17*). Nevertheless, the question does have a positive solution.

Two-slope planar theorem. *Every finite planar lattice with maximum up-degree and down-degree two has a planar, two-slope diagram.*

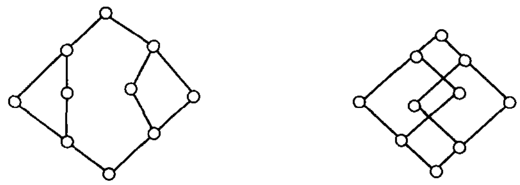


Fig. 17.

Proof. Let L be a finite planar lattice and let $L=\{x_1, x_2, \ldots, x_n\}$ be a dismantling sequence according to which each x_i is on the left boundary of a planar diagram of L-$\{x_1, x_2, \ldots, x_{i-1}\}$. We shall reconstruct L by a planar, two-slope diagram beginning with $K_1=\{x_{n-1}=b<t=x_n\}$. Suppose this edge is drawn at 45°. Suppose now that $K_{i+1}=\{x_n, x_{n-1}, \ldots, x_{i+1}\}$ has already been drawn by means of a planar, two-slope diagram. We show how to construct $K_i=K_{i+1}\cup\{x_i\}$ with a planar, two-slope diagram. If $\overline{x_i}\succ \underline{x_i}$ in K_{i+1} then we may partition the line segment already drawn joining them in K_i to accommodate x_i. Thus, suppose that $\overline{x_i}$ is not an upper cover of $\underline{x_i}$ in K_{i+1} although $\bar{x}_i>\underline{x_i}$ and both lie on the left boundary of K_i and of K_{i+1}. Let $\underline{x_i}\prec v\leqslant u\prec\overline{x_i}$ in K_i. According to the hypothesis $\underline{x_i}$ has no other upper cover and $\bar{x}_i$ has no other lower cover, in K_{i+1}; hence both v and u are on the left boundary of K_{i+1}. If the edge $\underline{x_i}$ to v has a 45° angle and u to $\overline{x_i}$, 135°, then we adjoin x_i on the left at 135° from $\underline{x_i}$ and at 45° to $\overline{x_i}$. Let us suppose then that the $\underline{x_i}$ to v edge, say, has a 135° angle. Then, as v is on the left boundary $\underline{x_i}$ is its unique lower cover (*Fig. 18*).

Our aim now is to transform the given planar, two-slope drawing of K_{i+1} to another planar, two-slope drawing in which only the angle of the edge from $\underline{x_i}$ to v is changed, to 135°, with the left boundary, in particular, of both drawings

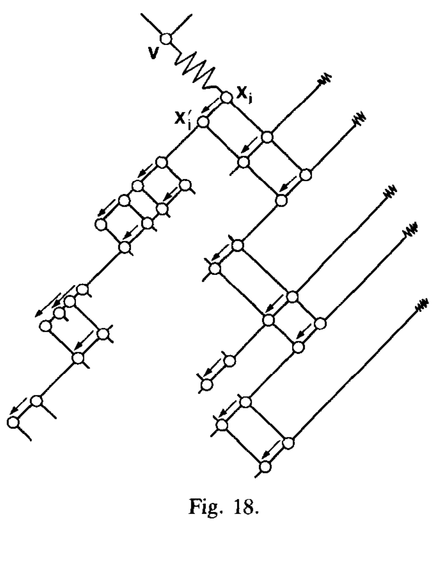


Fig. 18.

containing the same chain of elements. To this end consider the set A of elements $y \leq \underline{x}_i$ as in the original drawing of K_{i+1}. This "down set" has a top element a, a bottom b, a left boundary and a right boundary. The transformation is in two stages. The first is this. Sever the edge from $\underline{x}_i$ to v and shift A a unit distance along the 45° beam, thereby stretching any edges at 45° from elements on the right boundary of A to other elements of $L - A$. Of course, $L - A$ together with this transformed portion A' is not even a lattice for, in particular, v and $\underline{x}_i$ have no infimum. The second stage, however, fastens $\underline{x}_i$ to v again by shifting just the elements of A' which lie on the 45° line to $\underline{x}_i'$ along the 135° beam until they coincide with the 45° line to v. All edges of A' along 135° which meet any element moved are stretched accordingly and a 45° edge from $\underline{x}_i''$ to v is restored (see *Fig. 19*). What we have now is another planar, two-slope diagram of K_{i+1} with $\underline{x}_i$ located on the left boundary joined by a 45° edge to v. $\overline{x}_i$ is unchanged on the left boundary. If the edge from u to $\overline{x}_i$ is at 135° then we may, as before, complete the construction of K_i. If this edge is at 45° then the same two-stage transformation may be applied to the set of all elements $z \geq \overline{x}_i$, in order, finally to accommodate x_i. □

It seems reasonable too to expect a positive solution to the original "degree-slope" conjecture at least for planar lattices.

Conjecture III. For every positive integer $n \geq 3$, every finite planar lattice with maximum up-degree and down-degree n has a diagram with n slopes.

Again our current techniques for two slopes seem to shed no immediate light on this conjecture.

There is an apparently related problem concerning the representation of orders with two slopes. Obviously, the ordered set consisting of the direct product of two

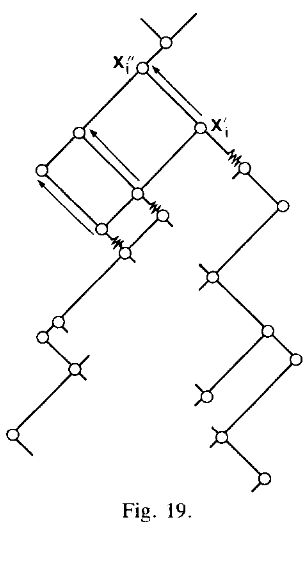
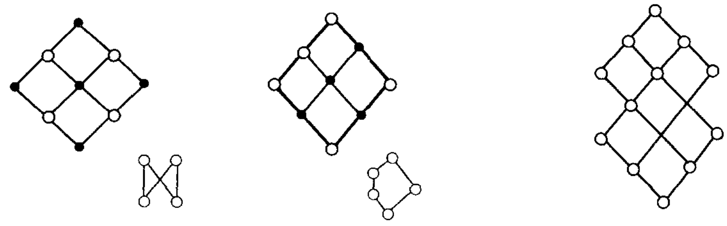


Fig. 19.



a b c

Fig. 20.

chains has a two-slope diagram as a "diamond-shaped grid" with the two chains orthogonal. We may ask about those ordered sets which can be order-embedded in the direct product of two chains and drawn with edges that follow the grid lines (without bends but possibly skipping grid vertices). For instance, the four-element ordered set illustrated in *Fig. 1a* can be order-embedded in the direct product of two chains but cannot, of course, be drawn to follow the grid lines, for this ordered set requires three slopes (see *Fig. 20a*).

On the other hand this ordered set is not a lattice. Actually, any lattice that can be order-embedded in the direct product of two chains must be planar (cf. [6], see *Fig. 20b*). And, as a matter of fact, the proof of the Two-Slope Planar Theorem makes clear that every finite planar lattice with maximum up-degree and down-degree two can be order-embedded in the direct product of two chains and can be drawn following the grid lines. However, a lattice which has a two-slope diagram need not be order-embeddable in the direct product of two chains, for that would imply that it is planar. The lattice illustrated in *Fig. 20c* is nonplanar and yet has a two-slope diagram.

4. Complexity of drawing two-slope diagrams

There remain these questions.

How can we decide whether a given lattice has a two-slope diagram?
If it does, how can we draw one?

According to our main result we must find a suitable colouring of the covering pairs. Once done we may draw a two-slope diagram efficiently using the procedure outlined in the proof. We do not, however, know whether the existence of this colouring can be efficiently decided.

Conjecture IV. The problem to decide whether a lattice has a two-slope diagram (and to draw a two-slope diagram) is NP-complete.

We have some promising evidence that, indeed, this problem is, as we believe, intractable. Take any lattice L and assign to each of its covering pairs of elements a *literal* (a Boolean variable or its negation) according to the following rule:

the literal assigned to $\{a, x\}$ is the negation of the literal assigned to $\{b, y\}$ if and only if there exists a sequence $\{a, x\} = e_1, \ldots, e_{2n} = \{b, y\}$ of covering pairs such that any e_i and e_{i+1} are of the form $e_i = \{c, z\}$ and $e_{i+1} = \{d, z\}$ where c and d are two upper covers or two lower covers of z.

Next identify all covering chains C whose endpoints are joined by at least one other covering chain. For any such C construct the Boolean formula f_C of the form

$$(\alpha_1 \vee \cdots \vee \alpha_n) \wedge (\neg\alpha_1 \vee \cdots \vee \neg a_n),$$

where $\alpha_1, \ldots, \alpha_n$ are literals assigned to the covering pairs in C. It is clear that a Boolean evaluation satisfies this formula if and only if not all covering pairs in C are given the same value. Let f be the conjunction of f_C over all covering chains C described above. Call f a formula associated to the lattice. It follows that a Boolean evaluation satisfies f if and only if the Boolean values given by it to all covering pairs yield a two-colouring described by the Two-Slope Theorem. Hence the colouring problem and the problem of finding a two-slope diagram reduces to "satisfiability" for the class of Boolean formulae associated with lattices. It is well known that satisfiability for arbitrary formulae is an NP-complete problem. We conjecture that the same is true for this restricted class of formulae too.

There is at least one important exception to this discouraging view of efficiency and that is for planar lattices. It is well known, and there are several techniques available, polynomial in the number of vertices, (cf. [7]), to decide whether a lattice is planar and, to construct a planar representation, if it is. This is because planar lattices are precisely the lattices od dimension two. Once a single planar representation of a lattice is presented, the proof of our Two-Slope Planar Theorem ensures an efficient way to construct a two-slope planar diagram of it. Thus for planar lattices at least the two-slope theory seems complete and satisfying. The problem to decide whether an ordered set has the dimension three or more is on the other hand, NP-complete ([10], cf. [7]).

Note added in proof Recently, J. Czyzowicz settled Conjecture I in the affirmative, for all n, (Lattice diagrams with few slopes, to appear in J. Combin. Theory Ser. A) and Conjecture III in the negative by providing a counterexample for $n = 3$ (Planar lattices and the slope problem, Ars Combinatoria 27 (1989) 101–112).

Acknowledgement

We are especially grateful to Jorge Urrutia for many stimulating discussions about this problem.

References

[1] M. Ajtai, On a class of finite lattices, Period Math. Hungar, 4 (1973) 217–220.
[2] K.A. Baker, P.C. Fishburn and F.S. Roberts, Partial orders of dimension 2, Networks, 2 (1971) 11–28.
[3] D. Duffus, Private communication (1984).
[4] D. Duffus and I. Rival, Crowns in dismantlable partially ordered sets, Colloq. Math. Soc. Janos Bolyai, 18 (1978) 271–292.
[5] D. Kelly and I. Rival, Crowns, fences, and dismantlable lattices, Canad. J. Math. 26 (1974) 1257–1271.
[6] D. Kelly and I. Rival, Planar lattices, Canad. J. Math., 27 (1975) 636–665.
[7] D. Kelly and W. T. Trotter, Jr., Dimension theory for ordered sets, in Ordered Sets (I. Rival, Ed.) (D. Reidel, 1982) 171–211.
[8] I. Rival, Lattices with doubly irreducible elements, Canad. Math. Bull., 17 (1974) 91–95.
[9] B. Sands, Problem 2.7 in Graphs and Order (I. Rival, Ed.) (D. Reidel, 1985) 531.
[10] M. Yannakakis, The complexity of the partial order dimension problem, SIAM J. Alg. Disc. Math., 3 (1981) 351–358.